Closed Form Expressions for Linear MIMO System Responses and Solutions of the Lyapunov Equation

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Abstract—In this paper, closed form expressions of linear MIMO system responses are presented. These expressions make use of matrix polynomial formulations of e^{tA} . They depend on the eigenvalues of A, the characteristic polynomial of A, as well as the partial fraction expansion coefficients of the corresponding unity numerator transfer function. The general partial fraction coefficients may be calculated using a computationally efficient recursive formula that has been derived in the context of SISO system responses. The final expressions are presented in a form that enhances the understanding of linear systems as such and emphasizes efficient computational implementation and the resulting time complexity. Finally, a closed form solution of the Lypunov equation, the Gramian, is presented.

Keywords: Matrix Exponential, Partial Fraction Expansion Coefficients, Linear Continuous Time MIMO System Responses, Lyapunov equation, Gramian

I. INTRODUCTION

There exists an extensive literature within the fields of ordinary differential equations, difference equations, matrix theory and Laplace transforms on closed formed expressions. The majority of such results, however, predates the computer era, and is not presented in a form that has onus on efficient algorithmic implementations. This fact, somewhat surprisingly, is still reflected in modern textbooks, e.g., in control theory, in the area of signals and systems as well as mathematics. In these textbooks, the corresponding types of results are presented in a restrictive setting, with little or no attention to how they could be implemented in general algorithms. Computer algorithms that have been developed over recent decades, e.g., within control theory and mathematics, on the other hand, are often based on general approaches to numerical solutions of ordinary differential equations and linear equations that do not make specific use of the structure that lies in the closed form expressions.

Thus naturally, much attention has been given to numerical methods during the past decades with the rapid development of fast computers. Those generally provide approximate solutions which are often applicable to large systems, see e.g., [1] regarding the computation of matrix exponentials and [2] and [3] regarding the solutions of Lyapunov equations. Despite the effectiveness and advantages of such numerical methods, closed form time domain solutions nevertheless

provide direct, easy and accurate computation for small to midsize systems. Further, closed form solutions open a window of opportunities definitely worth exploring, generally in the control area for the design of controllers and model reduction, both in their own right for small to midsize systems and by combining them with numerical methods for large systems.

Closed form continuous time transfer function expressions (SISO case) were derived in [4] and extended to the case of repeated eigenvalues in [5]. The closed form lends itself well to computation and analysis of transfer function responses and opens up many new interesting applications, e.g., solving for optimal zero locations by minimizing transient responses [4]; tracking a given reference step response in [6], and addressing the model reduction problem by $\mathcal{L}_2/\mathcal{H}_2$ minimization in [7]. The closed form expressions were further used in the direct computation of coefficients for PID and generalized PID controllers in [8] and [9].

It is of interest to extend the results obtained for SISO systems to the MIMO case. Naturally, the results obtained for SISO systems can be used directly for MIMO systems in the transfer function matrix form. In the case of MIMO systems in the state space form, the computation of the matrix exponential e^{tA} becomes of interest. Many different approaches have been proposed to compute the matrix exponential based, e.g. on eigenvector expansions of the matrix A, rational approximations to the exponential function and exact polynomial representations making use of the Cayley Hamilton theorem, see e.g. [10], [11] and references therein. It should be noted that for large A matrices, the computation of the matrix exponential itself is not computationally attractive and may be plagued by roundoff error[11]. However, in the case of MIMO responses, the central computational task is to calculate the vector $e^{tA}b$ for a given vector b and a given matrix A. For this task, the computation can be arranged into a recursive procedure that lends itself to efficient implementation. These procedures can be derived in many different ways, making e.g. use of properties of (generalized) Vandermonde matrices and their inverses, interpolation polynomials and inverse Laplace transforms. Here we choose to emphasize the connection with the Laplace transforms, highlighting the potential benefits of the procedure by applying it to the task of calculating Gramians and solving the standard Lyapunov equation. This

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approach can e.g. be contrasted with that used in Matlab's lyap which transforms the corresponding system matrices to the Schur form, computes the solution of the resulting triangular system and transforms the solution back[12].

While some of the basic ideas presented in this paper are certainly not new, cf e.g., [13] and [14], care has been taken to formulate them in a framework that can be readily implemented in a computational environment like Matlab, in an efficient manner. The analogy with the SISO case is also emphasized, since these forms have already proved to lend themselves nicely to a number of control applications in that case. It should further be emphasized that the main motivation behind this work is to provide another tool in the linear systems toolbox, to be used along with methods that have already been developed, e.g. numerical approaches, indeed these may support each other in further development.

In the next section the computational framework is first introduced. Then a proof of the general matrix polynomial form for e^{tA} that is the basis of our approach is presented. This is followed by results of applying this form to the problem of calculating general linear MIMO system responses, as well as closed form Gramians and solutions to the Lyapunov equation. Finally, numerical examples illustrating these approaches is presented.

II. A MATRIX POLYNOMIAL EXPRESSION OF e^{tA}

Consider the general state space representation of MIMO systems in the minimal form given by

$$\dot{x} = Ax + Bu
 y = Cx + Du$$
(1)

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{r \times n}$ and $D \in \mathbb{R}^{r \times p}$. Assume the matrix A has the characteristic equation

$$\det(sI - A) = \sum_{i=0}^{n} a_i s^i = (s - \lambda_1)^{d_1} (s - \lambda_2)^{d_2} \cdots (s - \lambda_{\nu})^{d_{\nu}},$$
(2)

where $a_n=1$ and $\lambda_i, i=1,\ldots,\nu$ are the eigenvalues of A. The corresponding Jordan matrix is given by

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{\nu} \end{bmatrix}$$
 (3)

with the diagonal blocks

$$J_{i} = \begin{bmatrix} \lambda_{i} & 1 & \cdots & \cdots & 0 \\ 0 & \lambda_{i} & 1 & & \vdots \\ 0 & 0 & \lambda_{i} & 1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda_{i} \end{bmatrix}, \tag{4}$$

each a $d_i \times d_i$ matrix.

Now, consider a basic rational function with a unity numerator:

$$F_{b}(s) = \frac{1}{s^{n} + a_{n-1}s^{n-1} + \dots + a_{0}}$$

$$= \frac{1}{(s - \lambda_{1})^{d_{1}}(s - \lambda_{2})^{d_{2}} \cdots (s - \lambda_{\nu})^{d_{\nu}}}$$

$$= \sum_{i=1}^{\nu} \sum_{i=1}^{d_{i}} \frac{\kappa_{ij}}{(s - \lambda_{i})^{j}},$$
(5)

where κ_{ij} are the basic partial fraction expansion coefficients

$$\kappa = \begin{bmatrix} \kappa_{11} & \cdots & \kappa_{1d_1} & \cdots & \kappa_{\nu 1} & \cdots & \kappa_{\nu d_{\nu}} \end{bmatrix}^T. \quad (6)$$

The term basic response refers here to the response of a transfer function containing only poles and a unity numerator, i.e., the basic impulse response $y_b(t)$ is the solution

$$y_b^{(n)}(t) + a_{n-1}y_b^{(n-1)}(t) + \ldots + a_0y_b(t) = \delta(t),$$
 $t > 0.$ (7)

The basic response is then naturally given by

$$y_b(t) = \sum_{i=1}^{\nu} \sum_{j=1}^{d_i} \kappa_{ij} \frac{t^{(d_i - j)}}{(d_i - j)!} e^{\lambda_i t} = \kappa^T \mathcal{E}(t), \qquad t > 0.$$
(8)

The unity numerator partial fraction coefficients κ are easily computed recursively as in [5], i.e.,

$$\kappa_{ij} = \begin{cases}
\prod_{q=1, q \neq i}^{\nu} \frac{1}{(\lambda_i - \lambda_q)^{d_q}}, \ j = d_i \\
\sum_{q=1}^{d_i - j} \frac{\kappa_{i(j+q)}(-1)^q}{d_i - j} \times \\
\sum_{p=1, p \neq i}^{\nu} \frac{d_p}{(\lambda_i - \lambda_p)^q}, \ j = d_i - 1, \dots, 1.
\end{cases}$$
(9)

Finally, define the corresponding linearly independent basis functions, contained in the $n \times 1$ vector,

$$\mathcal{E}(t) = \begin{bmatrix} \mathcal{E}_{1}(t) \\ \mathcal{E}_{2}(t) \\ \vdots \\ \mathcal{E}_{\nu}(t) \end{bmatrix}$$
 where
$$\begin{bmatrix} e^{\lambda_{i}t} \\ \frac{d}{d\lambda_{i}}e^{\lambda_{i}t} \\ \vdots \\ \frac{1}{(d_{i}-1)!}\frac{d^{d_{i}-1}}{d\lambda_{i}^{d_{i}-1}}e^{\lambda_{i}t} \end{bmatrix} = \begin{bmatrix} e^{\lambda_{i}t} \\ te^{\lambda_{i}t} \\ \vdots \\ \frac{t^{(d_{i}-1)}}{(d_{i}-1)!}e^{\lambda_{i}t} \end{bmatrix}.$$
 (11)

Theorem 1: We can express e^{tA} in the matrix polynomial

Theorem 1: We can express e^{tA} in the matrix polynomial form

$$e^{tA} = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i} a_{n-i+j} A^{j} \right) (J^{n-i-1} \kappa)^{T} \mathcal{E}(t)$$
(12)

$$= \sum_{i=0}^{n-1} (\sum_{j=1}^{n-i} a_{n-j+1} (J^{n-j-i} \kappa)^{T} \mathcal{E}(t)) A^{i},$$
(13)

$$= \sum_{i=0}^{n-1} (\gamma_{i}^{T} \mathcal{E}(t)) A^{i},$$
(14)

$$= \sum_{i=0}^{n-1} \alpha_{i}(t) A^{i},$$
(15)

where γ_i , $i = 0, 1, \dots, n-1$, are $n \times 1$ vectors, which can be computed recursively as

$$\gamma_{n-1} = \kappa, \quad \gamma_{n-k-1} = J\gamma_{n-k} + a_{n-k}\kappa, \quad k = 1, 2, \dots, n-1.$$
(16)

Note that $\alpha_i(t) = \gamma_i^T \mathcal{E}(t)$ in (14) and (15) is a scalar function of t, in accordance with the Cayley Hamilton theorem.

Proof: The matrix exponential can be written as the inverse Laplace transform of

$$e^{tA} = \mathcal{L}^{-1} \left\{ \frac{1}{\det(sI - A)} Adj(sI - A) \right\}. \tag{17}$$

We can write the adjoint matrix as

$$Adj(sI - A) = s^{n-1}I + s^{n-2}(A + a_{n-1}I) \qquad (\lambda_i)^m \quad C_1^m(\lambda_i)^m + s^{n-3}(A^2 + a_{n-1}A + a_{n-2}I) + \dots + (A^{n-1} + a_{n-1}A^{n-2} + \dots + a_1I) \qquad \text{where } C_q^p = \begin{pmatrix} p \\ q \end{pmatrix} \text{ and } + A^{n-1}(s^2 + a_{n-1}s + a_{n-2}) + \dots + I(s^{n-1} + a_{n-1}s^{n-2} + \dots + a_1).$$

$$(18)$$

We have for a unity numerator transfer function that

$$\mathcal{L}^{-1}\left\{\frac{1}{\det(sI-A)}\right\} = y_b(t) = \kappa^T \mathcal{E}(t). \tag{19}$$

It is easily seen that

$$y_b^{(k)}(t) = (J^k \kappa)^T \mathcal{E}(t). \tag{20}$$

Similarly,

$$\mathcal{L}^{-1}\left\{\frac{s^{j}}{det(sI-A)}I\right\} = y_{b}^{(k)}(t)I = I(J^{j}\kappa)^{T}\mathcal{E}(t), \quad j \ge 0$$
(21)

Then (12) immediately follows using the first half of (18). Using the second half of (18), we obtain (13) leading directly to (14) and (15) where

$$\gamma_i = \sum_{j=1}^{n-i} a_{n-j+1}(J^{n-j-i}\kappa), \quad i = 0, 1, \dots, n-1, \quad (22)$$

are $n \times 1$ vectors, which can be computed recursively as given by (16). *Q.e.d.*

Remark 1: The expression (14) can also be derived from the fact that e^{tA} can be expressed as p(A;t) where $p(\lambda;t)$ denotes the unique interpolation polynomial of degree n-1that interpolates $e^{t\lambda}$, viewed as a function of λ with t being constant, at the points $\lambda_1, \ldots, \lambda_{\nu}$ in the sense that

$$\frac{d^{j}}{d\lambda^{j}}p(\lambda;t)\bigg|_{\lambda=\lambda_{i}} = \frac{d^{j}}{d\lambda^{j}}e^{t\lambda}\bigg|_{\lambda=\lambda_{i}} \qquad j=0,1,\ldots,d_{i}-1.$$
(23)

Thus, if we express this interpolation polynomial in the form

$$p(\lambda;t) = \sum_{i=0}^{n-1} c_i(t)\lambda^i,$$
(24)

then (23) implies that, see e.g., [15]

$$V^T c(t) = \mathcal{E}(t) \tag{25}$$

where V is the confluent Vandermonde matrix given by

$$V = \begin{bmatrix} V_1 & V_2 & \cdots & V_{\nu} \end{bmatrix} \tag{26}$$

and

$$V_{i} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ (\lambda_{i}) & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_{i})^{m-1} & C_{1}^{m-1}(\lambda_{i})^{m-2} & \cdots & C_{d_{i-1}}^{m-1}(\lambda_{i})^{m-d_{i}} \\ (\lambda_{i})^{m} & C_{1}^{m}(\lambda_{i})^{m-1} & \cdots & C_{d_{i-1}}^{m}(\lambda_{i})^{m-d_{i}+1} \end{bmatrix}$$

where
$$C_q^p = \left(egin{array}{c} p \ q \end{array}
ight)$$
 and

$$c(t) = \begin{bmatrix} c_0(t) & \cdots & c_{n-1}(t) \end{bmatrix}^T. \tag{27}$$

Hence

$$c_i(t) = v_i^T \mathcal{E}(t), \tag{28}$$

where v_i denotes the *i*-th column of V^{-1} . We finally have

$$v_{n-1} = \kappa, \quad v_{n-k-1} = Jv_{n-k} + a_{n-k}\kappa, \quad k = 1, 2, \dots, n-1,$$
(29)

i.e., $\gamma_i = v_i$ and $c_i(t) = \gamma_i^T \mathcal{E}(t)$ (cf. [16] which also includes an alternative derivation of the formulation (14)-(16)). By introducting other equivalent formulations for $p(\lambda;t)$ such as the Lagrange form or the Newton form, we can derive other alternative formulations for e^{tA} . The expression derived for the Newton form[17] is in many respects comparable to the expression (14) and has the advantage that it does not rely on explicit knowledge of the coefficients a_i , whereas the Lagrange form is less attractive from a computational viewpoint.

Remark 2: The evaluation of the $n \times n$ matrices A^i , i = $2, \ldots, n-1$ requires $\mathcal{O}(n^4)$ operations and is as such not computationally attractive, see [10]. However, if we are to evaluate

$$e^{tA}b = \sum_{i=0}^{n-1} (A^i b)(\gamma_i^T \mathcal{E}(t))$$
 (30)

for an *n*-vector b, and we evaluate the vectors $A^{i}b$ recursively, then the number of operations reduces to $\mathcal{O}(n^3)$. Furthermore, if we are to evaluate $e^{tA}b$ repeatedly for different values of t, we can first calculate the matrix

$$Q = \sum_{i=0}^{n-1} (A^i b) (\gamma_i)^T$$
 (31)

in $\mathcal{O}(n^3)$ operations after which each evaluation of $e^{tA}b$ only requires $\mathcal{O}(n^2)$ operations, in addition to the evaluation of $\mathcal{E}(t)$.

Remark 3: Let $y_b(t)$ denote the "basic response" function whose transform $F_b(s)$ is given by (5). Hence it follows from (20) and (16) that:

$$\begin{bmatrix} \gamma_0^T \\ \gamma_1^T \\ \vdots \\ \gamma_{n-2}^T \\ \gamma_{n-1}^T \end{bmatrix} \mathcal{E}(t) = \begin{bmatrix} a_1 & \cdots & a_{n-1} & 1 \\ \vdots & \cdot & \cdot & 0 \\ a_{n-1} & \cdot & \cdot & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} Y_b(t) \quad (32)$$

where

$$Y_{b}(t) = \begin{bmatrix} y_{b}(t) \\ y'_{b}(t) \\ \vdots \\ y_{b}^{(n-2)}(t) \\ y_{b}^{(n-1)}(t) \end{bmatrix}$$
(33)

Remark 4: We have similarly the following expressions:

$$\frac{d^k}{dt^k}e^{tA} = \sum_{i=0}^{n-1} ((J^k \gamma_i)^T \mathcal{E}(t)) A^i$$
 (34)

and provided none of the eigenvalues are zero

$$\int \cdots \int e^{tA} (dt)^k = \sum_{i=0}^{n-1} ((J^{-k}\gamma_i)^T \mathcal{E}(t)) A^i.$$
 (35)

Remark 5: Expressions (12)-(16) can be extended to any matrix function f(At) that can be defined in terms of a Taylor series, and hence by the Cayley-Hamilton theorem in terms of a matrix polynomial of degree n-1. The only change in the expressions is $\mathcal{E}(t)$ that now becomes the vector

$$\begin{bmatrix} f(\lambda t) \\ \frac{d}{dt}f(\lambda t) \\ \vdots \\ \frac{d^{n-1}}{dt^{n-1}}f(\lambda t) \end{bmatrix}$$
 (36)

III. MIMO SYSTEM RESPONSES

Corollary 1: Assume that the input u(t) is a piecewise continuous function. Then the MIMO system response can be expressed as

$$y(t) = \sum_{i=0}^{n-1} (\gamma_i^T \mathcal{E}(t)) (CA^i x(0)) \\ + \sum_{j=1}^{p} \sum_{i=0}^{n-1} \gamma_i^T \int_0^t u_j(\tau) \mathcal{E}(t-\tau) d\tau (CA^i B_{.j}) \\ + Du(t)$$

Proof:

$$x(t) = e^{tA}x(0) + \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$= \sum_{i=0}^{n-1} (\gamma_{i}^{T}\mathcal{E}(t))(A^{i}x(0))$$

$$+ \int_{0}^{t} \sum_{i=0}^{n-1} \gamma_{i}^{T}\mathcal{E}(t-\tau)A^{i}Bu(\tau)d\tau$$

$$= \sum_{i=0}^{n-1} (\gamma_{i}^{T}\mathcal{E}(t))(A^{i}x(0))$$

$$+ \sum_{j=1}^{p} \sum_{i=0}^{n-1} \gamma_{i}^{T} \int_{0}^{t} u_{j}(\tau)\mathcal{E}(t-\tau)d\tau(A^{i}B_{.j})$$
(38)

Q.e.d.

Remark 6: If we wish to calculate y(t) repeatedly for different values of t, we can as before first calculate the matrices

$$Q_0 = \sum_{i=0}^{n-1} (CA^i x(0)) \gamma_i^T$$
 (39)

and

$$Q_j = \sum_{i=0}^{n-1} (CA^i B_{.j}) \gamma_i^T, \quad j = 1, 2, \dots, p$$
 (40)

with $\mathcal{O}(n^4)$ operations for each matrix and subsequently calculate x(t) for each t with $\mathcal{O}(pn^2)$ operations in addition to having to evaluate $\mathcal{E}(t)$ and $\int_0^t u_j(\tau)\mathcal{E}(t-\tau)d\tau$, $j=1,2,\ldots p$.

Remark 7: The impulse response of a MIMO system is given by

$$y_I(t) = Ce^{tA}B + D\delta(t), \tag{41}$$

which noting that the term $\gamma_i^T \mathcal{E}(t)$ in (14) is a scalar, can be expressed as

$$y_I(t) = \sum_{i=0}^{n-1} (\gamma_i^T \mathcal{E}(t)) C A^i B + D\delta(t).$$
 (42)

Note that in this case the evaluation of the $r \times p$ matrix $Ce^{tA}B$ will require $\mathcal{O}(\min(r,p)n^3)$ operations if we calculate the Markov parameters CA^iB recursively in the appropriate order.

Remark 8: This expression can be contrasted with that obtained by making use of the closed form expressions for SISO systems[5]. In particular, it can be show that the scalar impulse response corresponding to the k-th input and the l-th output can be expressed as

$$y_{I,k,l}(t) = \sum_{i=0}^{n-1} C_k A^i B_{\cdot l} \gamma_i^T \mathcal{E}(t) + D_{kl} \delta(t)$$
 (43)

$$= C_k \cdot \hat{\mathcal{B}}_l \mathcal{K}^T \mathcal{E}(t) + D_{kl} \delta(t), \tag{44}$$

where C_k denotes the k-th row vector of C and $B_{\cdot l}$ the l-th column vector of B. K, is a $n \times n$ matrix,

$$\mathcal{K} = \left[\begin{array}{ccc} \kappa & J\kappa & \cdots & J^{n-1}\kappa \end{array} \right] \tag{45}$$

which can be computed recursively from κ in $\mathcal{O}(n^2)$ operations and the column vectors of the matrix

$$\hat{\mathcal{B}}_l = \left[\begin{array}{ccc} \beta_{0,l} & \beta_{1,l} & \cdots & \beta_{n-1,l} \end{array} \right], \tag{46}$$

can be computed recursively from the column vectors of B for k = 1, 2, ..., n - 1:

$$\beta_{n-1,l} = B_{\cdot l}, \quad \beta_{n-k-1,l} = A\beta_{n-k,l} + a_{n-k}B_{\cdot l}.$$
 (47)

With this formulation the main computational task is that of calculating the matrices $\hat{\mathcal{B}}_j$, $j=1,2,\ldots,p$ from (47) which requires $\mathcal{O}(pn^3)$ operations. Thus, the two alternative expressions (42) and (44) are comparable from a computational point of view. Note that (44) correspond to the impulse response of a SISO system given by[5]

$$y_I(t) = (\mathcal{KB})^T \mathcal{E}(t), \qquad t > 0,$$
 (48)

where the numerator coefficients

$$\mathcal{B} = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix}^T \tag{49}$$

are given by

$$\mathcal{B} = C_k \cdot \hat{\mathcal{B}}_l = \begin{bmatrix} C_k \cdot (A^{n-1} + a_{n-1}A^{n-2} + \dots + a_2A + a_1I)B_{\cdot l} \\ \vdots \\ C_k \cdot (A^2 + a_{n-1}A + a_{n-2}I)B_{\cdot l} \\ C_k \cdot (A + a_{n-1}I)B_{\cdot l} \\ C_k \cdot B_{\cdot l} \end{bmatrix}$$
(50)

IV. SOLVING THE LYAPUNOV EQUATION

Consider the Lyapunov equation

$$AP + PA^H + BB^H = 0 (51)$$

and its solution given by the input Gramian.

Corollary 2: The input Gramian

$$P = \int_0^\infty e^{tA} B B^H e^{tA^H} dt \tag{52}$$

can be expressed as

$$P = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left(\gamma_i^T \int_0^\infty \mathcal{E}(t) \mathcal{E}(t)^H dt \bar{\gamma}_j \right) A^i B B^T \left(A^T \right)^j.$$
(53)

Proof: We have from Theorem 1 that

$$e^{tA}B = \sum_{i=0}^{n-1} \left(\gamma_i^T \mathcal{E}(t)\right) A^i B \tag{54}$$

and the result follows directly.

Q.e.d.

Remark 9: A dual result may be derived for the output Gramian and subsequently the Hankel singular values may be computed.

Remark 10: Calculating the matrices A^iB for i = $0, 1, \ldots, n-1$ recursively, the evaluation of P according to (53) requires $\mathcal{O}(pn^3)$ operations.

Remark 11: The (ρ, σ) -th element of the (k, j)-th subblock of $\int_0^\infty \mathcal{E}(t)\mathcal{E}(t)^H dt$, i.e., of the matrix $\int_0^\infty \mathcal{E}_k(t)\mathcal{E}_j(t)^H dt$ is given by $\frac{\binom{\rho+\sigma-2}{\rho-1}}{(-\lambda_k-\lambda_j)^{\rho+\sigma-1}}$. Alternatively, we can make use of (32) and express

$$P = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left(\tilde{a}_{i+1}^T \int_0^\infty Y_b(t) Y_b(t)^T dt \tilde{a}_{j+1} \right) A^i B B^T \left(A^T \right)^j,$$
(55)

where \tilde{a}_i denotes the *i*-th column vector of the matrix on the right hand side of (32). Here we note that the matrix $\int_0^\infty Y_b(t)Y_b(t)^T dt$ will have the following plaid like structure

$$\mathcal{A} = \begin{bmatrix}
A_{1} & 0 & -A_{2} & 0 & A_{3} & \cdots \\
0 & A_{2} & 0 & -A_{3} & 0 \\
-A_{2} & 0 & A_{3} & 0 & -A_{4} \\
0 & -A_{3} & 0 & A_{4} & 0
\end{bmatrix}, \quad \gamma_{\cdot 3} = \begin{bmatrix}
-0.2388 + 0.4553i \\
-0.2388 - 0.4553i \\
-1.5301 \\
2.0078
\end{bmatrix}, \quad \gamma_{\cdot 2} = \begin{bmatrix}
-0.5750 + 0.0484i \\
-0.5750 - 0.0484i \\
-0.5750 - 0.0484i \\
-0.1119 \\
1.2619
\end{bmatrix}, \quad \gamma_{\cdot 1} = \begin{bmatrix}
-0.2511 - 0.1184i \\
-0.2511 + 0.1184i \\
-0.2511 + 0.1184i
\end{bmatrix}, \quad \gamma_{\cdot 0} = \begin{bmatrix}
-0.0018 - 0.0009i \\
-0.0018 + 0.0009i \\
-0.0100 \\
1.0136
\end{bmatrix}.$$
(64)

where

$$\mathcal{A}_{i} = \int_{0}^{\infty} \left(y_{b}^{(i-1)}(t) \right)^{2} dt$$

$$= (J^{i-1}\kappa)^{T} \int_{0}^{\infty} \mathcal{E}(t) \mathcal{E}^{H}(t) dt \overline{J^{i-1}\kappa}.$$
(57)

This follows in turn from the fact that

$$y_b^{(i)}(0) = 0,$$
 $i = 0, 1, \dots, n - 2,$ $y_b^{(n-1)}(0) = 1$ $\lim_{t \to \infty} y_b^{(i)}(t) = 0,$ $i = 0, 1, \dots, n - 1.$ (58)

The coefficients A_i , i = 1, ..., n can alternatively be expressed directly in terms of a_i , i = 0, 1, ..., n, but we omit the details of that here.

V. EXAMPLES

A. System properties

We will use a well known MIMO model of a jet (see e.g. Matlab's Control Toolbox help) in order to illustrate the application of some of the formulae presented above. The inputs to the system, u_1 and u_2 , symbolize the rudder and aileron deflections, respectively, in degrees. The outputs y_1 and y_2 represent the yaw rate and bank angle. The corresponding system matrices are given by

$$A = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.5980 & -0.1150 & -0.0318 & 0 \\ -3.0500 & 0.3880 & -0.4650 & 0 \\ 0 & 0.0805 & 1.0000 & 0 \end{bmatrix}, (59)$$

$$B = \begin{bmatrix} 0.0073 & 0\\ -0.4750 & 0.0077\\ 0.1530 & 0.1430\\ 0 & 0 \end{bmatrix}, \tag{60}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{61}$$

B. Impulse response

We start by finding the closed form expression of the system's inpulse response, i.e. compute $y_I(t)$ using (42). Then the basic response partial fraction coefficients are computed from (9)

$$\kappa = \begin{bmatrix} -0.2388 + 0.4553i & -0.2388 - 0.4553i & -1.5301 & 2.0078 \end{bmatrix}^{T},$$
(62)

and the corresponding $\mathcal{E}(t)$ vector is given by

$$\mathcal{E}(t) = \begin{bmatrix} e^{(-0.0329 + 0.9467i)t} \\ e^{(-0.0329 - 0.9467i)t} \\ e^{-0.5627t} \\ e^{-0.0073t} \end{bmatrix}.$$
 (63)

Now the γ matrix can be computed recursively as in (16)

$$\gamma_{\cdot 3} = \begin{bmatrix} -0.2388 + 0.4553i \\ -0.2388 - 0.4553i \\ -1.5301 \\ 2.0078 \end{bmatrix}, \quad \gamma_{\cdot 2} = \begin{bmatrix} -0.5750 + 0.0484i \\ -0.5750 - 0.0484i \\ -0.1119 \\ 1.2619 \end{bmatrix},$$

$$\gamma_{\cdot 1} = \begin{bmatrix} -0.2511 - 0.1184i \\ -0.2511 + 0.1184i \\ -1.3736 \\ 1.8759 \end{bmatrix}, \quad \gamma_{\cdot 0} = \begin{bmatrix} -0.0018 - 0.0009i \\ -0.0018 + 0.0009i \\ -0.0100 \\ 1.0136 \end{bmatrix}.$$
(64)

Now we have all the elements needed to calculate the impulse response as in (42), resulting in

$$\begin{array}{lll} y_{I(1,1)} & = & 2 \cdot e^{-0.0329t} \left[\; 0.0147 \cdot sin(0.9467t) - 0.1691 \cdot cos(0.9467t) \; \right] \\ & & -0.0254 \cdot e^{-0.5627*t} - 0.1114 \cdot e^{-0.0073*t}, \\ \\ y_{I(1,2)} & = & -2 \cdot e^{-0.0329t} \left[\; 0.0020 \cdot sin(0.9467t) + 0.0021 \cdot cos(0.9467t) \; \right] \\ & & +0.0029 \cdot e^{-0.5627t} + 0.0090 \cdot e^{-0.0073t}, \\ \\ y_{I(2,1)} & = & 2 \cdot e^{-0.0329t} \left[\; 0.4404 \cdot cos(0.9467t) + 0.6217 \cdot sin(0.9467t) \; \right] \\ & & +1.8722 \cdot e^{-0.5627t} - 2.7530 \cdot e^{-0.0073t}, \\ \\ y_{I(2,2)} & = & 2 \cdot e^{-0.0329t} \left[\; 0.0127 \cdot sin(0.9467t) - 0.0032 \cdot cos(0.9467t) \; \right] \\ & & -0.2150 \cdot e^{-0.5627t} + 0.2213 \cdot e^{-0.0073t}. \end{array}$$

C. Input Gramian

In order to compute directly the output Gramian, the solution to the Lyapunov matrix equation, we first compute

$$\int_{0}^{\infty} \mathcal{E}(t)\mathcal{E}(t)^{H} dt = \begin{bmatrix} 15.1812 & 0.0184 + 0.5275i & 0.4761 + 0.7568i & 0.0448 + 1.0545i \\ 0.0184 - 0.5275i & 15.1812 & 0.4761 - 0.7568i & 0.0448 - 1.0545i \\ 0.4761 - 0.7568i & 0.4761 + 0.7568i & 0.4887 & 1.7546 \\ 0.0448 - 1.0545i & 0.0448 + 1.0545i & 1.7546 & 68.7005 \end{bmatrix}$$

utilizing Remark 11. Now it is straightforward to compute the solution to the Lyapunov equation by using (53), calculating A^iB recursively, resulting in

$$P = \begin{bmatrix} 1.8663 & -0.0066 & -2.9371 & 8.0258 \\ -0.0066 & 1.7427 & -2.8782 & 18.3346 \\ -2.9371 & -2.8782 & 16.9103 & -1.4759 \\ 8.0258 & 18.3346 & -1.4759 & 524.8139 \end{bmatrix}.$$
(67)

We also get the same result using (55)-(57). The output Gramian Q can be computed in an analogous manner and hence the Hankel singular values can be computed based on closed form solutions for P and Q.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, closed form expressions of linear MIMO system responses were presented. For MIMO systems in the state space form, the closed form expressions make use of matrix polynomial formulations of e^{tA} and depend on the eigenvalues of A and the characteristic polynomial of A as well as the partial fraction coefficients of the corresponding unity numerator transfer function. The general partial fraction coefficients can be calculated using a computationally efficient recursive formula that has been derived in the context of SISO system responses. The final expressions are presented in a form that emphasizes efficient computational implementation and the resulting time complexity. The results are valid for real as well as complex eigenvalues, the eigenvalues may be repeated.

Alternatively, the results obtained for SISO systems can be used directly for MIMO systems by making use of the transfer function matrix form. We also present closed form expressions based on this approach. The two alternative forms turn out to be comparable from a computational point of view.

In addition to the closed form expressions in MIMO systems responses, a closed form solution to the Lyaponov equation, i.e., the Gramian, was presented. Subsequently, the Hankel singular values may be computed. Paralleling the SISO case, the impulse response is linear in the numerator coefficients of the transfer function matrix and may open up numerous possibilities in optimization of the numerator coefficients or the element zeros of MIMO systems. Such optimization can possibly be applied towards model reduction as well as control of MIMO systems.

The emphasis in this work has been on the derivation of computationally efficient formulations of closed form expressions. The short term motivation has simply been to provide another tool in the linear systems toolbox to be used along with methods that have already been developed based on numerical approaches. The aim is to develop criteria based on numerical efficiency and stability to aid in the choice of appropriate solution tools. It is finally to be hoped that such expressions may enhance the understanding of linear systems as such and provide new approaches to solving problems in optimization and control as has proved to be the case with SISO systems.

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